Title: Music to Math: Exploring the Relationships between Frequencies of Musical Tones

Brief Overview:

Each tone on a musical scale has a specific frequency. The frequencies of the twelve tones in an octave have an exponential relationship. After completing this unit, students will be able to look for specific qualities of an exponential relationship, will be able to use the calculators to determine that relationship, and use that relationship to predict other values.

Links to Standards:

- **Mathematics as Problem Solving**
  Students will collect and analyze data and use this to make predictions.

- **Mathematics as Communication**
  Students will be able to explain procedures and processes used in data collection and analysis. They will be able to demonstrate their acquisition of knowledge by making predictions.

- **Mathematics as Reasoning**
  Students will make conjectures about the relationship among data and use the data to justify this relationship.

- **Mathematical Connections**
  Students will discover the connections between mathematical functions, scientific analysis of sound waves, and musical scale structure.

- **Functions**
  Students will analyze a set of data points through various techniques to determine the function that is most representative.

Links to Maryland High School Mathematics Core Learning Goals

- **1.1.1:** The student will recognize, describe, and extend patterns and functional relationships that are expressed numerically, algebraically, and geometrically.

- **1.1.2:** The student will represent patterns and functional relationships in a table, as a graph or by mathematical expression.

- **1.1.4:** The student will describe the graph of a function which is not a straight line and discuss its appearance in terms of the basic concepts of maxima and minima (highs and lows), roots (zeros), limits (boundaries), rate of change, and continuity.

- **3.1.1:** The student will design and/or conduct an investigation that uses statistical methods to analyze data and communicate results.

- **3.2.1:** The student will make informed decisions and predictions based upon the result of simulations and data from research.

- **3.2.2:** The student will make predictions by finding and using a line of best fit and by using a given curve of best fit.

Grade/Level:

College Algebra, Pre-Calculus, or Advanced Algebra II

Prerequisite Knowledge:

Students should understand the following terms: frequency, period, amplitude, and cycle. They also should have working knowledge of the following skills:

- Inputting data into lists and using regression equations on a TI-82/83
- Analyzing functions for common differences between consecutive data points to determine linearity

Objectives:

Students will:

- calculate the frequency of a tone from a graph of its sound wave.
- determine the most appropriate function to represent the data.
- understand the properties of an exponential function.
- use the function to make predictions of the frequency of other tones.

Materials/Resources/Printed Materials:

- The SOUND program written for the TI-82[1]

For each student group:

- CBL and microphone/amplifier probe
- Link cable
- Keyboard or other instrument to create notes

For each student:

- TI-82/83
- Class activity worksheets

Development/Procedures:

- Students will use the SOUND program to calculate the frequencies of five consecutive notes on the scale (middle C, C#, D, D#, E).
- Students will input these frequencies into a list.
- Students will graph the frequencies, using an index value starting at C=0, C#=1,... for the x-value and the frequency for the y-value.
- Students will calculate and graph linear and exponential regressions for these data points.
- Predictions will be made on the frequency of two other notes (G and high C).
- The calculated frequencies will be added to the list with the appropriate indices and graphed along with the original data points and regression equations.
- Students will draw conclusions about which regression equation is better based on the graph of the data.
- Students will find the difference and the ratio between consecutive terms and use them to justify the conclusions drawn above.

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[1] CBL System Experiment Workbook (Texas Instruments Incorporated, 1994)

Performance Assessment:

A short assessment is provided.

Extension/Follow Up:

The following topics lend themselves to extension activities to this lesson.

- Harmonic Ratios: The frequencies that make up major thirds, fourths, fifths, and octaves occur in specific ratios.
- Noise: A discussion on what makes noise, such as a car horn or whistle, by examining its frequency (or often, frequencies).
- Discordant sounds: Most often, it is listening to two or more tones whose frequencies do not have a simple harmonic ratio.
- Flat/sharp tones: Why do notes that are flat or sharp sound so bad? Often, it is only a matter of a few Hz in the frequency of the tone that makes the difference.
- Non-Western music: Some African and Asian music is based on a different musical scale than the traditional Western scale. Do the notes on these scales conform to the exponential model?
- Distance from fret to bridge on a guitar: This distance also has an exponential relationship, only the distances get smaller. An example of a decreasing exponential function, $0 < b < 1$.
- Geometric sequences: The terms of an exponential function are a geometric sequence, much the same way the terms of a linear function are an arithmetic sequence. This would be an appropriate place to introduce geometric sequences to the class.
- Exponential curves outlined by strings of a grand piano or pipes of some pipe organs.

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MUSIC TO MATH Name ______________________________

I. Set up.

Verify that the calculator to be connected to the CBL contains the program SOUND. Connect this calculator to the CBL using the link cable. Connect the microphone to the CBL through port CH1. Make sure that your keyboard works properly. If you have different instrument settings on the keyboard, choose a tone that has very little vibrato.

II. Determine the frequency of middle C.

Run the SOUND program on the calculator. Make sure that the CBL is turned on and there are three dashes on its screen. Place the microphone sensor on the speaker. Play middle C. You will only need to play the note for a few seconds. If nothing registers, turn up the volume some and try again. You should have the graph of a sound wave on the calculator.

Use the trace key to determine the x-values at the first peak and the sixth peak. Subtract the x-value at the first peak from the x-value at the sixth peak. This will be the time in seconds for five cycles to occur. This should be a very small number. Record this number in the table below. Record all decimal places shown. Divide this number by five in order to determine the period for one cycle. Record this value in the space provided, again keeping all decimal places. Find the inverse of the period. This will be the frequency for middle C. Record this value to one decimal place in the table.

III. Determine the frequency of the next four consecutive notes.

Repeat the above procedure for C# (C-sharp), D, D# and E and record below.

Index	Note	Time (sec) for 5 cycles	Period (sec/cycle)	Frequency (cycles/sec)
0	C			
1	C#			
2	D			
3	D#			
4	E			

IV. Enter the data into the lists on your calculator and graph it.

Enter the index for each note into L_1. Enter the corresponding frequencies into L_2. Press [2nd] [Y=] to enter the "STAT PLOT" menu. Make sure that Plot2 and Plot3 are turned off. Select Plot1. Turn the plot on, select the scatter plot, Xlist as L_1, Ylist as L_2, and square as the mark. Select an appropriate window based on your x- and y-values. Press [GRAPH] to see the plot of the data. Adjust the size of the window if necessary. Record your window size below.

Window Used

Xmin __________ Xmax ____________

Ymin __________ Ymax ____________

Based on the way the graph looks, what type of relationship do you suspect that this has? Explain your reasoning. ____
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V. Perform regression analysis on data.

To perform a linear regression on the data, follow these instructions. Press [STAT], [->] to access the "CALC" menu, then select "LinReg". LinReg should appear on the home screen. It uses the defaults of L_1 and L_2. If your lists are not in L_1 and L_2, you must tell LinReg where your lists are, x-list first, separated by a comma. Press [ENTER] to calculate the regression equation.

To store this equation in Y_1, press [Y=] and clear Y_1, leaving the cursor there. Now press [VARS], select "Statistics", select the "EQ" menu and choose "RegEQ". This will store the equation in Y_1 to graph later.

To perform the exponential regression, repeat this procedure choosing "ExpReg" in place of "LinReg" and Y_2 instead of Y_1. Record the regression equations below. You may round to four decimal places for recording purposes, but be aware that for accuracy in later calculations you will need all the decimal places given to you by the calculator.

Linear Regression Equation: Y = ____

Exponential Regression Equation: Y = ____________________________________

Press the [GRAPH] key to see both regression equations along with the data. Increase the Xmax and Ymax of the viewing window to see how the graphs of the two regression equations diverge.

As you can see, both regression equations fit the current data very well. Which equation do you think will be a better predictor of other data points? ______________________________________

VI. Predict frequency of other notes.

On the standard musical scale, using the index we have started with, G would be note 7 and high C would be note 12. Using these indices as values for x, calculate the predicted frequency for G and C using both regression equations. Since we already have the complete equations stored in Y_1 and Y_2, we can use the table function of the calculator to accurately calculate these values. Press [2nd] [WINDOW] to get to the "TblSet" menu. Make sure that the "Indpnt:" setting is set to "Ask" and the "Depend:" is set to "Auto". Now press [2nd] [GRAPH] to see the table. You will have an empty table with columns labeled X, Y_1 and Y_2. Enter the values of 7 and 12 into the X column. The calculator will automatically determine the values of Y_1 and Y_2 for these x-values. Record these values in the table below.

Index	Note	Predicted Frequency using Linear Regression	Predicted Frequency using Exponential Regression
7	G		
12	C		

VII. Determine actual frequencies of G and C using CBL.

Repeat procedure in section II for G and high C. Record the values in the table below.

Index	Note	Time (sec) for 5 cycles	Period (sec/cycle)	Frequency (cycles/sec)
7	G			
12	C			

VIII. Analyze new data.

Add the frequencies calculated above and their corresponding indices to L_1 and L_2 and regraph along with the original points and the two regression equations.

Which regression equation is a better predictor of the actual frequency? ______________________

What type of relationship do the notes in a musical scale have? ______________________________

VIII. Analyze the data points to find a numerical relationship.

In a linear relationship, there is a common difference between two consecutive data points. This difference is called the slope. In this last section we will try to determine if a similar relationship exists for exponential relationships.

Currently, we have data in L_1 and L_2. Go to the lists by pressing [STAT] and selecting "Edit". Move the cursor to the **heading** L_3 and copy L_2 into it by letting L3 = L2. Move the cursor to the first element in L_3. Press the [DEL] key to delete it. This will move every element in L_3 up one space. Move the cursor to the first empty cell in L_3 and enter 1. Move the cursor to the **heading** L_4 and let $L_4 = L_3 - L_2$. Copy the **first four** values of L2, L3 and L4 into the table below. L_4 now represents the difference between the frequencies of consecutive notes which are stored in L_2.

L2	L3	L4	L5

Is there a common difference between consecutive terms? If so, what is it? __________________

Is this a linear relationship? Explain. ____

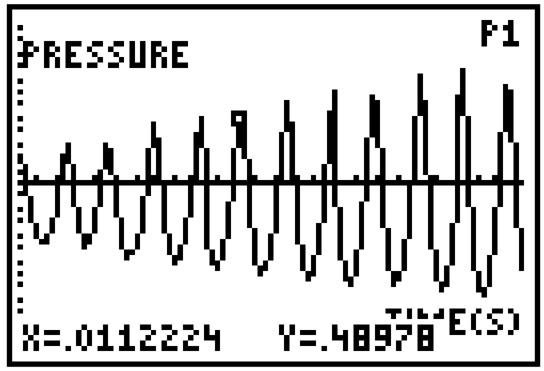
Now move the cursor to the **heading** L_5 and let $L_5 = L_3 / L_2$. Copy these values into L5 in the table above.

What do you notice about the values in L5? ____

How does this value relate to the consecutive terms in L2? _________________________________

ASSESSMENT

1. The picture at right shows the graph of a sound wave. Using the X value given and the location of the trace cursor on the graph, calculate the frequency of this wave. (Assume the x-value at the first peak is zero.)



2. Given the regression equation Y = 261.7064 * 1.0594 ^ X, predict the frequency of the note with the index of 9 on the scale that starts at C = 0.

3. The scale of musical tones goes C - C# - D - D# - E - F - F# - G - G# - A - A# - B - C. For a particular instrument, low C has a frequency of 262 cycles/sec and consecutive tones have a common ratio of 1.06. Which note would have a frequency of approximately 351 cycles/sec?

TEACHER NOTES

- An electronic keyboard was used to create the tones used in this lab experiment, and we found that the flute or violin instrument mode produced the highest quality sound wave. Ideally, tuning forks would work best because of the clarity of the tone, but the teacher would need five consecutive tuning forks (middle C, C#, D, D#, E) as well as G and upper C. This could also be done with another musical instrument that will produce a clear tone, but the teacher should verify the clarity of the tone before conducting the experiment. A guitar was attempted but it proved difficult.

- The microphone needs to be very close to the sound source--in our case we put it directly on the speaker of the keyboard with the microphone sensor facing down.

- The teacher will need to make sure that each calculator that will be connected to a CBL is loaded with the program SOUND (this program is provided in the software that comes with GRAPH LINK). The microphone probe connects to CH1 port on the CBL. The program has two quirks that you should be aware of:

    1. The graph that is produced has the label "Pressure" on the y-axis. This is because the microphone amplifier measures the pressure that a sound wave exerts on air molecules over a given time period.

    2. After exiting the graph using the clear button, the home screen will read "READY..." but you will need to hit the enter key to restart the program. This will give the message "PRESS ENTER TO ARM THE SENSOR." which will reset the CBL to the data collection mode. Make sure that the CBL screen displays three dashes ("---").

- The sound wave may appear to grow in amplitude. The amplitude of the wave represents its volume. The electronic keyboard does not produce instantaneous volume but takes a few milliseconds to reach the peak volume; this change is not distinguishable to the human ear. However, the sensor begins recording instantaneously and can detect this change. The graph represents only about the first 1/40th of a second of the sound wave, which is why the change is visible on the graph even though we do not hear it.

- Ideally students will be working in groups of 3-5, depending on availability of equipment. This may work better with one instrument and an overhead calculator connected to a CBL. Student volunteers could operate the equipment with all students calculating frequencies and manipulating data individually on their own calculator.

- Depending on the background of the class it might be appropriate to begin with a brief discussion on properties of waves - frequency, period, amplitude and cycle.

- The teacher should verify that each student group has approximately the same frequency for the first note, just to make sure that everyone is doing the experiment and the calculations properly.

- You may find that when, after plotting the regression equations at the end of section V, it might be appropriate to lead a class discussion on which equation is more appropriate. You could also have students perform a quadratic regression, then extend the lower bound of the graph to show that this is an inappropriate function because the frequency would have to increase as the tone got lower.

- The table below lists the names of the notes in order and the standard frequencies associated with them. You will likely find a slight variation from this list in your instrument. The keyboard that we used was found to be slightly high on every tone (ex. middle C = 265 Hz). The tuning forks we came across were at a slightly lower frequency (ex. middle C = 256 Hz).

Note	Frequency
C	262
C#	277
D	294
D#	311
E	330
F	349
F#	370
G	392
G#	415
A	440
A#	466
B	494
C	524

- The regression equation used in #2 of the assessment was generated using the above table of values. Because every instrument is tuned slightly different, answers may very. However, the values should still have an exponential relationship and should be similar to the values listed here.

- In part VIII, you should only focus on the first four values because after that you will be dealing with non-consecutive tones.

- **If the students have already been introduced to arithmetic and geometric sequences, you should be able to relate the exponential relationship to a geometric sequence.**